Generalized Involutive Bases and Their Induced Free Resolutions

Matthias Orth[1]

[1]Universität Kassel, Kassel, Germany

27 July 2022

Table of Contents

Table of Contents

Polynomials and Ideals

- $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$: polynomial ring
- $\mathcal{T}$: set of terms $\mathbf{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$.
- Polynomial: Finite sum of monomials $c\mathbf{x}^\mu$ $(c \in \mathbb{K})$.
- $\deg(\mathbf{x}^\mu) = \sum_{i=1}^n \mu_i$: degree of the term $\mathbf{x}^\mu$
- *Generating set* or *basis* of ideal $\mathcal{I} \trianglelefteq \mathcal{R}$: $G = \{g_1, \ldots, g_m\} \subseteq \mathcal{I}$ with

$$\langle G \rangle := \left\{ \sum h_i g_i \mid h_1, \ldots, h_m \in \mathcal{R} \right\} = \mathcal{I}$$

- *Homogeneous polynomial of degree d*: $f = \sum c_i t_i$ with $\deg(t_i) = d$.
- *Homogeneous ideal*: Ideal generated by homogeneous polynomials
- If $G = \{t_1, \ldots, t_m\} \subset \mathcal{T} \rightsquigarrow \langle G \rangle$ *monomial ideal*.

We work with homogeneous and monomial ideals.

Term Orderings

- $\prec$: fixed term ordering
- $\mathrm{lt}(f)$: leading term of $0 \neq f \in \mathcal{R}$
- $\mathrm{lt}(\mathcal{I}) = \langle \mathrm{lt}(f) \mid f \in \mathcal{I} \rangle$: leading ideal of $0 \neq \mathcal{I} \trianglelefteq \mathcal{R}$
- Via term orderings, can use monomial ideals to study general ideals.

Gröbner Bases

Definition

$G = \{g_1, \ldots, g_m\} \subset \mathcal{I}$ Gröbner basis of ideal $\mathcal{I} \trianglelefteq \mathcal{R}$, if $\langle \mathrm{lt}(g_1), \ldots, \mathrm{lt}(g_m) \rangle = \mathrm{lt}(\mathcal{I})$.

Definition

Let $f, g \in \mathcal{R}$ be two monic polynomials. Their S-polynomial is $S(f, g) = (\mathsf{lcm}(\mathrm{lt}(f), \mathrm{lt}(g))/\mathrm{lt}(f)) \cdot f - (\mathsf{lcm}(\mathrm{lt}(f), \mathrm{lt}(g))/\mathrm{lt}(g)) \cdot g$.

Theorem (Buchberger 1965)

One can compute a Gröbner basis of any given ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ by producing new leading terms via S-polynomials.

Involutive Divisions and Bases

Involutive division L: Restricted division relation for terms.

- Given set $H \subset \mathcal{T}$ and $\mathbf{x}^\mu \in H$, assigns multiplicative variables $L(\mathbf{x}^\mu, H) \subseteq \{x_1, \dots, x_n\}$.
- The involutive cones $\mathcal{C}_L(\mathbf{x}^\mu, H) = \{\mathbf{x}^\nu \mid \mathbf{x}^\nu/\mathbf{x}^\mu \in \mathbb{K}[L(\mathbf{x}^\mu, H)]\}$ intersect only trivially as $\mathbf{x}^\mu$ varies in H.
- Filter axiom: If $H \supseteq H'$, then $L(\mathbf{x}^\mu, H') \supseteq L(\mathbf{x}^\mu, H)$.
- Polynomial set $H \subset \mathcal{I} \trianglelefteq \mathcal{R}$ *L-involutive basis* of ideal $\mathcal{I}$, if

$$\bigsqcup_{\mathbf{x}^\mu \in \mathrm{lt}(H)} \mathcal{C}_L(\mathbf{x}^\mu, \mathrm{lt}(H)) = \mathrm{lt}(\mathcal{I}).$$

Every involutive basis is a Gröbner basis.

Pommaret Bases and Quasi-stable Ideals

Definition

Pommaret division P: term $\mathbf{x}^\mu$ with $\mu = (0, \dots, 0, \mu_k, \dots, \mu_n)$ and $\mu_k > 0$ $\rightsquigarrow P(\mathbf{x}^\mu) = \{x_1, \dots, x_k\}$. Define $\mathsf{cls}(\mathbf{x}^\mu) = \min\{k \mid \mu_k > 0\}$.

Remark

Not every monomial ideal $\mathcal{I}$ has a finite Pommaret basis. A monomial ideal with finite Pommaret basis is called quasi-stable.

Example: Pommaret Basis

Example

- Consider $\mathcal{I} = \langle x^2, y^2 \rangle \trianglelefteq \mathbb{K}[x, y]$, we have $\mathrm{cls}(x^2) = 1$, $\mathrm{cls}(y^2) = 2$.
- We obtain a Pommaret basis by adding the generator $x^2 y$.

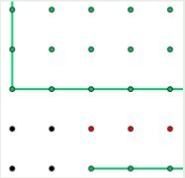


Figure: The ideal $\mathcal{I}$ and the Pommaret cones of its generators

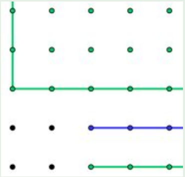


Figure: The Pommaret cone of $x^2 y$ added.

Janet Division

- Let $U = \{u_1, \ldots, u_m\} \subset \mathcal{T}$ be a set of terms. For $d_i, \ldots, d_n \in \mathbb{Z}_{\geq 0}$, define a *Janet class* of U:

$$U_{[d_i,\ldots,d_n]} := \left\{u \in U \mid \deg_j(u) = d_j,\ i \leq j \leq n\right\} \subseteq U\ .$$

  Note that $U_{[]} = U$.
- A variable x_i is *Janet multiplicative* for $u \in U_{[d_{i+1},\ldots,d_n]}$, if $\deg_i(u) = \max\{\deg_i(v) \mid v \in U_{[d_{i+1},\ldots,d_n]}\}$.
- Every monomial ideal has a Janet basis.

Table of Contents

Modules and Syzygies

- $\mathcal{R}^m$: Free $\mathcal{R}$-module of *rank m* with elements $\mathbf{f} = (f_1, \ldots, f_m)^T$.
- Write $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$, 1 at position i.
- The *terms* of $\mathcal{R}^m$ are $\mathbf{x}^\mu \cdot \mathbf{e}_i$ with $\mathbf{x}^\mu \in \mathcal{T}$ and $i \in \{1, \ldots, m\}$.

Definition

The *syzygy module* of $G = \{g_1, \ldots, g_m\} \subseteq \mathcal{R}$ is defined as:

$$\mathrm{Syz}(G) = \left\{ (f_1, \ldots, f_m)^T \in \mathcal{R}^m \mid \sum_{i=1}^{m} f_i \cdot g_i = 0 \right\} \subseteq \mathcal{R}^m.$$

Syzygies of Gröbner Bases

$G = \{g_1, \dots, g_m\}$ Gröbner basis, $S(g_i, g_j)$ S-polynomial: Write $S(g_i, g_j) = \sum h_k g_k$ with suitable $h_k \in \mathcal{R} \rightsquigarrow$ syzygy $\mathbf{S}_{ij} \in \operatorname{Syz}(G)$.

Theorem (Schreyer 1980)

$G = \{g_1, \dots, g_m\}$ *Gröbner basis* $\Rightarrow \{\mathbf{S}_{ij} \mid 1 \leq i < j \leq m\} \subset \operatorname{Syz}(G)$ *Gröbner basis of* $\operatorname{Syz}(G)$ *with respect to suitable ordering on terms of* $\mathcal{R}^m$.

Application

By iteration, obtain *free resolution* of $\mathcal{I} = \langle G \rangle$.

Free Resolutions

Definition

Free resolution **F** of $\mathcal{I}$:
Finitely generated free $\mathcal{R}$-modules $F_0, F_1, \ldots$, $\mathcal{R}$-linear maps $\delta_0, \delta_1, \ldots$:

$$\mathbf{F}: \quad 0 \xrightarrow{0} F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} \mathcal{I} \to 0,$$

with $\text{im}(\delta_0) = \mathcal{I}$ and $\text{im}(\delta_{m+1}) = \ker(\delta_m)$ for all m.
Differential of **F**: Collection $\{\delta_0, \ldots, \delta_n\}$.

Remark

δ_m is represented by matrix D_m with entries in $\mathcal{R}$, and $D_m \cdot D_{m+1} = 0$.

Definition

Resolution **F** is *minimal*: $\Longleftrightarrow$ no constant terms $\neq 0$ in matrices D_m.

Importance of Minimal Free Resolutions

- Minimal free resolution of $\mathcal{I}$ contains important invariants of $\mathcal{I}$: *Castelnuovo-Mumford regularity* and *projective dimension*
- For arbitrary monomial ideals $\mathcal{I}$, general formula for the minimal free resolution is a long-standing open problem (Kaplansky, 1960s)

Table of Contents

Overview

	in $\mathcal{R}$	in $\mathcal{R}/\mathcal{I}$
Gröbner bases	Buchberger (1965)	e.g. Spear (1977), La Scala & Stillman (1998)
Involutive Bases	Janet (1920), Gerdt & Blinkov (1998), Seiler (2009)	this thesis
Involutive-like bases	Gerdt & Blinkov (2005) (Janet-like), this thesis (Pommaret-like)	this thesis
Marked bases	Cioffi & Roggero (2011), Ceria et al. (2015)	this thesis

Acknowledgements

This thesis includes joint work with:

- Amir Hashemi and Werner Seiler (involutive and involutive-like bases),
- Cristina Bertone, Francesca Cioffi, and Werner Seiler (marked bases).

Table of Contents

Cones and Complementary Decompositions

- Cone: set of terms $\mathcal{C}_Y(t) = \mathcal{T}_Y \cdot t$, where $t \in \mathcal{T}$ vertex and $\mathcal{T}_Y$ set of terms of $\mathbb{K}[Y]$, with $Y \subseteq \{x_1, \ldots, x_n\}$.
- Monomial ideal $\mathcal{I}$: Involutive bases decompose $\mathcal{I} \cap \mathcal{T}$ into cones.
- Cone decomposition of $\mathcal{T} \setminus \mathcal{I}$: Complementary decomposition (C.D.).

Definition (not new)

Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal. The *Hilbert function* of $\mathcal{I}$ is:
$\mathrm{HF}_{\mathcal{I}} : \mathbb{N}_0 \to \mathbb{N}_0,\ d \mapsto |\{t \in \mathcal{T} : \deg(t) = d \wedge t \notin \mathcal{I}\}|$.

For $d \gg 0$, $\mathrm{HF}_{\mathcal{I}}(d) = \mathrm{HP}_{\mathcal{I}}(d)$, with $\mathrm{HP}_{\mathcal{I}} \in \mathbb{Q}[x]$ *Hilbert polynomial* of $\mathcal{I}$.

Application

C. D. of $\mathcal{I}$ easily yields its Hilbert function and polynomial.

Janet Tree

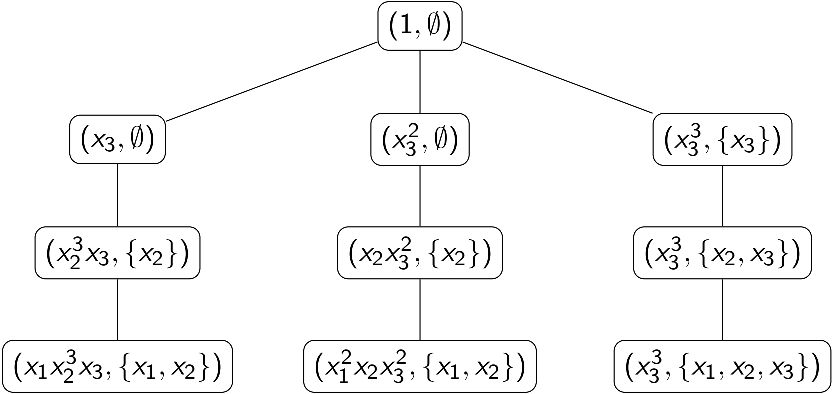
Aim

Apply (generalized) involutive bases to the efficient computation of complementary decompositions.

Given $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$, the Janet tree of U contains nodes with labels (t, V), where...

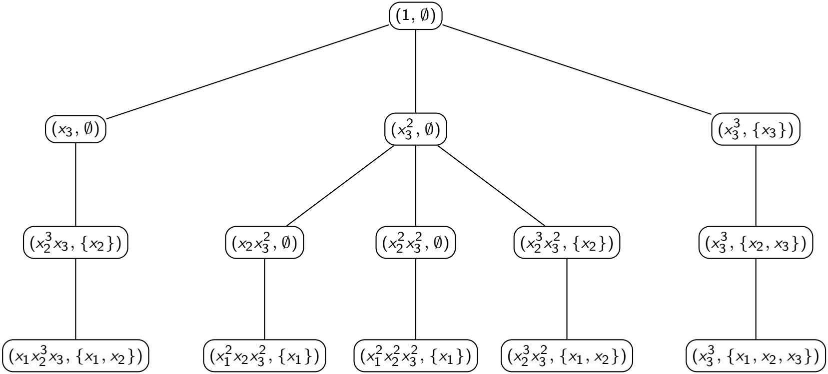
1. ... $t \in \mathcal{T}$ encodes a Janet class of U,
2. ... V is the set of variables multiplicative for each term in this class.

Janet Tree of Term Set



Janet tree of $U = \{x_1 x_2^2 x_3,\ x_1^2 x_2 x_3^2,\ x_3^3\} \subset \mathbb{K}[x_1, x_2, x_3]$.

Janet Tree of Janet Basis



Janet tree of Janet basis

$$H = \{x_1 x_2^3 x_3,\ x_1^2 x_2 x_3^2,\ x_1^2 x_2^2 x_3^2,\ x_2^3 x_3^2,\ x_3^3\} \subset \mathbb{K}[x_1, x_2, x_3].$$

Janet-like Division

The *Janet-like division* was discovered by Gerdt and Blinkov in 2005.

- Let $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$.
- If x_i is Janet non-multiplicative for $u \in U$, the power $x_i^{k_i}$ with

$$k_i = \min\left\{\deg_i(v) - \deg_i(u) \mid v, u \in U_{[d_{i+1},\ldots,d_n]}, \deg_i(v) > \deg_i(u)\right\}$$

  is called a non-multiplicative power of u.
- The set $\mathrm{NMP}(u, U)$ collects all non-multiplicative powers of $u \in U$.
- J-multipliers for $u \in U$: terms not divisible by any $x_i^{k_i} \in \mathrm{NMP}(u, U)$.
- $u \in U$ is Janet-like divisor of $w \in \mathcal{T}$, if w/u is a J-multiplier for u.

Janet-like Tree

Given $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$, the Janet-like tree of U contains nodes with labels (t, V, M), where...

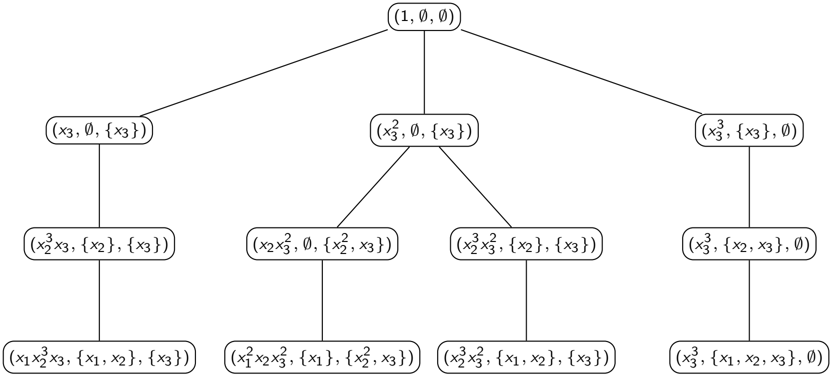
1. ... $t \in \mathcal{T}$ encodes a Janet class of U,
2. ... V is the set of variables multiplicative for each term in this class.

Janet-like Tree

Given $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$, the Janet-like tree of U contains nodes with labels (t, V, M), where...

1. ... $t \in \mathcal{T}$ encodes a Janet class of U,
2. ... V is the set of variables multiplicative for each term in this class.
3. ... M is the set of non-multiplicative powers common to all terms in this class.

Janet-like Tree of Janet-like Basis (1)



Janet-like tree of Janet-like basis

$$B = \{x_1 x_2^3 x_3,\ x_1^2 x_2 x_3^2,\ x_2^3 x_3^2,\ x_3^3\} \subset \mathbb{K}[x_1, x_2, x_3].$$

Compressed Complementary Decomposition

Algorithm 1: Complementary decomposition (from Janet-like basis)

Data: Janet-like tree JLT of Janet-like basis of monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$

Result: Finite complementary decomposition D of $\mathcal{I}$

begin

  $D \longleftarrow \emptyset$

  for $k = n, \ldots, 1$ **do**

    foreach *node* $(\mathbf{x}^\rho, V', M')$ *at level* $k+1$ *in* JLT **do**

      let $(x_k^m \mathbf{x}^\rho, V, M)$ be the leftmost child of $(\mathbf{x}^\rho, V', M')$

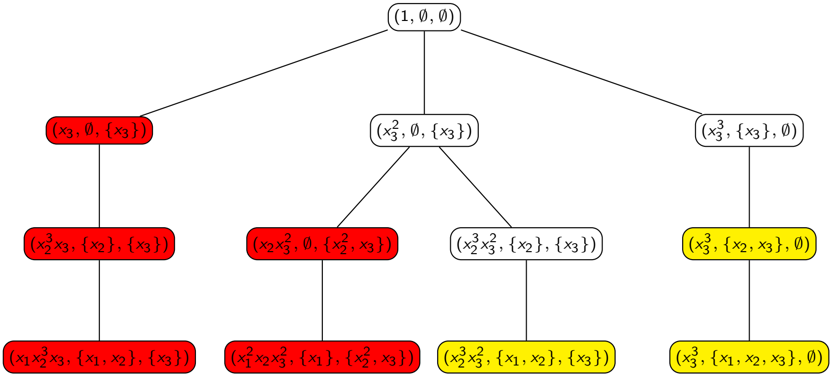
      if $m > 0$ **then**

        $N \longleftarrow \{x_1, \ldots, x_{k-1}\} \cup V'$

        $D \longleftarrow D \cup \left\{(\mathbf{x}^\rho \mathbf{x}^\mu, N) : \mathbf{x}^\mu | x_k^{m-1} \prod_{x_a^{h_a} \in M'} x_a^{h_a - 1}\right\}$

  return D

Janet-like Tree of Janet-like Basis (2)



Nodes in the Janet-like tree of $B = \{x_1x_2^3x_3,\ x_1^2x_2x_3^2,\ x_2^3x_3^2,\ x_3^3\}$ that contribute cones to the C.D. of $\mathcal{I} = \langle B \rangle$. (In red.)

Example Complementary Decomposition

Example

Using algorithm and Janet-like tree, obtain following complementary decomposition for $\mathcal{I} = \langle x_1 x_2^3 x_3,\ x_1^2 x_2 x_3^2,\ x_2^3 x_3^2,\ x_3^3 \rangle$. Cones are given as pairs of vertex and set of multiplicative variables.

$$
\begin{aligned}
&(1, \{x_1, x_2\}),\\
&(x_3, \{x_1\}), (x_2 x_3, \{x_1\}), (x_2^2 x_3, \{x_1\}),\\
&(x_2^3 x_3, \{x_2\}),\\
&(x_3^2, \{x_1\}),\\
&(x_2 x_3^2, \emptyset), (x_2^2 x_3^2, \emptyset), (x_1 x_2 x_3^2, \emptyset), (x_1 x_2^2 x_3^2, \emptyset).
\end{aligned}
$$

The cones in each line are encoded in a single node of the Janet-like tree.

Hironaka's Construction

Let $k \in \{0, 1, \ldots, n\}$; projections $\mathrm{pr}_k : \mathcal{T} \to \mathcal{T},\ t \mapsto t|_{x_1 = \cdots = x_k = 1}$.
For $t \in \mathcal{T}$, define the cone $\mathcal{C}_k(t) = \mathcal{C}_{\{x_1, \ldots, x_k\}}(t)$.

Construction (Hironaka 1977)

$\mathcal{I} \trianglelefteq \mathcal{P}$ monomial ideal. Define $\overline{N}_k(\mathcal{I}) := \left(\mathcal{C}_{\{k+1\}}(\mathrm{pr}_{k+1}(\mathcal{I}) \cap \mathcal{T})\right) \setminus \mathrm{pr}_k(\mathcal{I})$. Then, a complementary decomposition of $\mathcal{I}$ is given by $\mathcal{T} \setminus \mathcal{I} = \bigsqcup_{k=0}^{n-1} \mathcal{C}_k(\overline{N}_k(\mathcal{I}))$, where $\mathcal{C}_k(\overline{N}_k(\mathcal{I})) = \bigcup_{s \in \overline{N}_k(\mathcal{I})} \mathcal{C}_k(s)$.

Theorem

Hironaka's construction yields a finite decomposition if and only if the input monomial ideal is quasi-stable. Janet-like trees can be used to efficiently compute this decomposition.

Recursive Quasi-Stability Test

$U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$, with x_n-degrees $\lambda_0 < \lambda_1 < \cdots < \lambda_\ell = \alpha$.
For $0 \leq i \leq \ell$, $U_{\lambda_i} \subseteq U$ subset of terms t with $\deg_n(t) = \lambda_i$;
$U'_{\lambda_i} = \left\{ t|_{x_n=1} \mid t \in U_{\lambda_i} \right\} \subset \mathbb{K}[x_1, \ldots, x_{n-1}]$.

Theorem

U is a Pommaret basis, if and only if:

1. *For each degree λ_i, the set U'_{λ_i} is a Pommaret basis.*
2. *For each i, we have $U'_{\lambda_i} \subset \langle U'_{\lambda_i+1} \rangle$,*
3. *We have $U \cap \mathbb{K}[x_n] = x_n^\alpha$.*

The ideal $\langle U \rangle$ is quasi-stable, if and only if:

1. *For each $i \leq \ell$, the ideal $\left\langle \bigcup_{j=0}^{i} U'_{\lambda_j} \right\rangle \subseteq \mathbb{K}[x_1, \ldots, x_{n-1}]$ is quasi-stable,*
2. *We have $U \cap \mathbb{K}[x_n] \neq \emptyset$.*

Sparse Transformations to Quasi-stable Position

Definition (not new)

A polynomial ideal $\mathcal{I}$ is in *quasi-stable position*, if $\mathrm{lt}(\mathcal{I})$ is quasi-stable.

Applications

1. Deterministic approach for reaching quasi-stable position.
2. Sparse linear changes of variables used.

Example

$\langle x_5x_6, x_4x_6, x_4x_5, x_3x_5, x_2x_5, x_3x_4, x_2x_4, x_2x_3, x_1x_3, x_1x_2\rangle \trianglelefteq \mathbb{K}[x_1, \ldots, x_6]$:
Algorithm based on recursive quasi-stability check makes the following linear coordinate changes to reach quasi-stable position:
$x_5 \mapsto x_5 + x_6$, $x_4 \mapsto x_4 + x_5$, $x_3 \leftrightarrow x_4$, $x_2 \mapsto x_2 + x_4$, $x_2 \leftrightarrow x_3$, $x_1 \mapsto x_1 + x_3$.

Further Recursive Criteria

Found similar recursive criteria for:

1. Janet bases (based on a result due to Janet)
2. Janet-like bases
3. Minimality of Janet(-like) bases

Application

Novel algorithm for the minimization of a given Janet basis.

Table of Contents

Involutive-like Divisions and Pommaret-like Division

Definition

Involutive-like division L on $\mathcal{T}$, given finite set $U \subset \mathcal{T}$ and $u \in U$, assigns:

- the *non-multiplicative powers* $\mathrm{NMP}_L(u, U)$ of $u \in U$
- the set of *L-non-multipliers* $\bar{L}(u, U) = \langle \mathrm{NMP}_L(u, U) \rangle$
- The set of *L-multipliers* $L(u, U)$, $\mathcal{T} \setminus \bar{L}(u, U)$.
- For any $u \in U$, the *involutive cone* $\mathcal{C}_L(u, U) = u \cdot L(u, U)$.

The cones must intersect only trivially. No filter axiom assumed.

Definition

The *Pommaret-like division* P assigns $\mathrm{NMP}_P(u, U)$ as follows:

- All Janet-like non-multiplicative powers $x_a^{p_a}$ with $a > \mathrm{cls}(u)$.
- The Janet multiplicative variables x_b with $b > \mathrm{cls}(u)$.

Some Properties of Involutive-Like Divisions

Theorem

1. *A monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is quasi-stable, if and only if it possesses a finite Pommaret-like basis.*
2. *From a Pommaret-like basis U of the monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$, one can obtain a Pommaret basis U' of $\mathcal{I}$ as follows:*

$$U' = \left\{ t \cdot \mathbf{x}^{\mu} : t \in U \wedge \mathbf{x}^{\mu} \mid \prod_{x_a^{p_a} \in \mathrm{NMP}_P(t,U)} x_a^{p_a - 1} \right\} .$$

3. *Any minimal Janet-like basis is Pommaret-like autoreduced.*
4. *The minimal Janet-like basis of a quasi-stable monomial ideal $\mathcal{I}$ is also a Pommaret-like basis of $\mathcal{I}$.*

H Pommaret-like basis, $h \in H$, $x_a^{p_a} \in \mathrm{NMP}_P(h, H) \rightsquigarrow$ syzygies induced by Pommaret-like reduction of $x_a^{p_a} \cdot h$ form Pommaret-like basis of $\mathrm{Syz}(H)$.

Theorem

Let $\mathcal{I} \trianglelefteq \mathcal{R}$ quasi-stable ideal generated by minimal Pommaret-like basis $H \subset \mathcal{I} \cap \mathcal{T}$. If H is the minimal generating set of $\mathcal{I}$ and for each non-multiplicative power $x_j^{p_j}$ of $t \in H$, the term $(t/x_{\mathrm{cls}(t)})x_j^{p_j} \in \mathcal{I}$, then the free resolution of $\mathcal{I}$ induced by H is minimal.

Remark

Module basis elements of F_m in free resolution are of the form $\mathbf{e}_{\alpha,\mathbf{x}^\mu}$ with $h_\alpha \in H$, $\mathbf{x}^\mu$ squarefree, $\deg(\mathbf{x}^\mu) = m$, and $\mathrm{cls}(\mathbf{x}^\mu) > \mathrm{cls}(h_\alpha)$.

Explicit Differential Formula

$H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ minimal Pommaret-like basis of $\mathcal{I} = \langle H \rangle$.

Definition

For $h_\alpha \in H$, $x_a^{p_a} \in \mathrm{NMP}(h_\alpha, H)$, $\exists_1\ h_\beta$ with $x_a^{p_a} \cdot h_\alpha \in \mathcal{C}_P(h_\beta, H)$.
Write $\Delta(\alpha, a) = \beta$, and $t_{\alpha,a} = (x_a^{p_a} \cdot h_\alpha)/h_\beta$.

Under mild assumptions on Δ:

Theorem

Let $\mathcal{I}$ be Δ-commuting. Write $\mathrm{NMP}(h_\alpha, H) = \{x_j^{p_j} \mid j > \mathrm{cls}(h_\alpha)\}$.
Differential δ given by: $\delta(\mathbf{e}_{\alpha,1}) = h_\alpha$, and, for $\deg(\mathbf{x}^\mu) > 0$,

$$\delta(\mathbf{e}_{\alpha,\mathbf{x}^\mu}) = \sum_{x_j \in \mathrm{supp}(\mathbf{x}^\mu)} (-1)^{\mathrm{sgn}(x_j, \mathrm{supp}(\mathbf{x}^\mu))} \cdot \left(x_j^{p_j} \mathbf{e}_{\alpha, \mathbf{x}^\mu / x_j} - t_{\alpha,j} \mathbf{e}_{\Delta(\alpha,j), \mathbf{x}^\mu / x_j} \right).$$

Example Resolution (1)

Remark

Generalizes resolution formula for stable monomial ideals (Eliahou & Kervaire, 1990), and formula for quasi-stable ideals (Seiler, 2009).

Example

Consider $\mathcal{I} = \langle xy, y^3, xz, y^2z, z^2 \rangle$. The induced free resolution is:

$$D_0 = \begin{pmatrix} xy & y^3 & xz & y^2z & z^2 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} y^2 & z & 0 & 0 & 0 & 0 \\ -x & 0 & z & 0 & 0 & 0 \\ 0 & -y & 0 & y^2 & z & 0 \\ 0 & 0 & -y & -x & 0 & z \\ 0 & 0 & 0 & 0 & -x & -y^2 \end{pmatrix}, \; D_2 = \begin{pmatrix} z & 0 \\ -y^2 & 0 \\ x & 0 \\ -y & z \\ 0 & -y^2 \\ 0 & x \end{pmatrix}.$$

Example Resolution (2)

Example

Again, let $H = \{xy, y^3, xz, y^2z, z^2\}$. Write $h_\alpha = xy$, $h_\beta = y^3$, $h_\gamma = xz$, $h_\delta = y^2z$, and $h_\epsilon = z^2$. Differential formula yields, e.g.:

$$\begin{aligned} \delta(\mathbf{e}_{\alpha,yz}) &= z\mathbf{e}_{\alpha,y} \quad - y\mathbf{e}_{\gamma,y} - y^2\mathbf{e}_{\alpha,z} \quad + x\mathbf{e}_{\beta,z} \\ \delta(\mathbf{e}_{\gamma,yz}) &= z\mathbf{e}_{\gamma,y} \qquad\qquad\quad - y^2\mathbf{e}_{\gamma,z} \quad + x\mathbf{e}_{\delta,z}. \end{aligned}$$

P-graph of H encodes the function Δ for H.

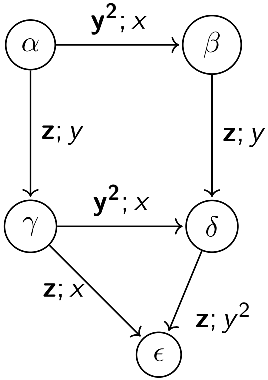


Figure: P-graph of the Pommaret-like basis H.

Table of Contents

Quotient Ring

We work in the quotient ring $\mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is an ideal.
Let $f \in \mathcal{R}$ be a polynomial.

- $[f] = [f]_{\mathcal{I}} = f + \mathcal{I}$: element of $\mathcal{R}/\mathcal{I}$
- $\mathrm{NF}_G(f)$: normal form of f with respect to a Gröbner basis G of $\mathcal{I}$: remainder of f after division by G.

Relative Gröbner Bases

- Ideal $\hat{\mathcal{J}} \trianglelefteq \mathcal{R}/\mathcal{I} \rightsquigarrow \hat{\mathcal{J}} = \mathcal{J}/\mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$.
- $[h] \in \mathcal{R}/\mathcal{I}$, G Gröbner basis of $\mathcal{I} \rightsquigarrow$ define $\mathrm{lt}([h]) := \mathrm{lt}(\mathsf{NF}_G(h))$.

Definition

$H = \{h_1, \ldots, h_k\} \subset \mathcal{J}$ Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ if

$$\langle \mathrm{lt}(H) \rangle + \mathrm{lt}(\mathcal{I}) = \mathrm{lt}(\mathcal{J}).$$

Proposition

H Gröbner basis of $\mathcal{J}$ $\Longrightarrow$
$\{\mathsf{NF}_G(h) \mid h \in H \setminus \mathcal{I}\}$ *Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

Relative Schreyer Theorem

Given: Ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ and $F \subset \mathcal{R} \setminus \mathcal{I}$.

- Can produce new leading terms by:
  1. S-polynomials among the elements of F
  2. annihilating leading terms modulo $\mathrm{lt}(\mathcal{I}) \rightsquigarrow$ A-polynomial
- S-polynomials and A-polynomials $\rightsquigarrow$ S-syzygies and A-syzygies

Theorem

H Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I} \Longrightarrow$
$\{S\text{–}syzygies\} \cup \{A\text{–}syzygies\}$ Gröbner basis of $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([h] \mid h \in H)$

Remark

The induced free resolution over $\mathcal{R}/\mathcal{I}$ is, in general, of infinite length.

Relative Involutive Divisions and Bases

Definition

$\mathcal{I}$ monomial ideal. Involutive division L on $\mathcal{T} \setminus \mathcal{I}$ relative to $\mathcal{I}$:
For $\mathbf{x}^\mu \in H \subset \mathcal{T} \setminus \mathcal{I}$: Multiplicative variables $L(\mathbf{x}^\mu, H) \rightsquigarrow$
Relative involutive cones $\mathcal{C}_{L,\mathcal{I}}(\mathbf{x}^\mu, H) = \mathcal{C}_L(\mathbf{x}^\mu, H) \setminus \mathcal{I}$.

Proposition

L classical involutive division, $\mathcal{I}$ monomial ideal, $\mathbf{x}^\mu \in H$, $H \cap \mathcal{I} = \emptyset$. Relative involutive division $L_\mathcal{I}$ induced by L:

$$x_i \in L_\mathcal{I}(\mathbf{x}^\mu, H) \iff x_i \in L(\mathbf{x}^\mu, H) \ \vee \ x_i \cdot \mathbf{x}^\mu \in \mathcal{I}.$$

Relative Involutive Divisions and Bases

Definition

$\mathcal{I}$ monomial ideal. Involutive division L on $\mathcal{T} \setminus \mathcal{I}$ relative to $\mathcal{I}$:
For $\mathbf{x}^\mu \in H \subset \mathcal{T} \setminus \mathcal{I}$: Multiplicative variables $L(\mathbf{x}^\mu, H) \rightsquigarrow$
Relative involutive cones $\mathcal{C}_{L,\mathcal{I}}(\mathbf{x}^\mu, H) = \mathcal{C}_L(\mathbf{x}^\mu, H) \setminus \mathcal{I}$.

Proposition

L classical involutive division, $\mathcal{I}$ monomial ideal, $\mathbf{x}^\mu \in H$, $H \cap \mathcal{I} = \emptyset$.
Relative involutive division $L_\mathcal{I}$ induced by L:

$$x_i \in L_\mathcal{I}(\mathbf{x}^\mu, H) \iff x_i \in L(\mathbf{x}^\mu, H) \ \vee \ x_i \cdot \mathbf{x}^\mu \in \mathcal{I}.$$

Definition

$H = \{h_1, \ldots, h_k\} \subset \mathcal{J} \setminus \mathcal{I}$ L-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, if

$$\bigsqcup_{h \in H} \mathcal{C}_{L,\mathcal{I}}(\mathrm{lt}(h), \mathrm{lt}(H)) = \mathrm{lt}(\mathcal{J}) \setminus \mathrm{lt}(\mathcal{I}).$$

Results on Relative Involutive Bases

- Algorithm for computing (inclusion-wise minimal) relative Janet bases.
- If a relative Pommaret basis exists, can compute minimal such basis.
- Pommaret basis for $\mathcal{J}$ relative to $\mathcal{I}$ exists $\iff$ $\mathrm{lt}(\mathcal{J})$ is *quasi-stable relative to* $\mathrm{lt}(\mathcal{I})$
- For $\mathcal{I} \subseteq \mathcal{J}$, can transform $\mathcal{J}$ to quasi-stable position relative to $\mathcal{I}$ with sparse linear changes of variables.

Free Resolution by Minimal Relative Pommaret Bases

H classical or relative involutive basis, $h \in H$.
Non-multiplicative prolongation: $x_i \cdot h$ where $x_i \notin L(\mathrm{lm}(h), \mathrm{lm}(H))$.

Theorem

$\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$, G minimal Pommaret basis of $\mathcal{I}$, H minimal Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$. For each $h \in H$ take:

1. *Syzygies $\mathbf{S}_{h,x_i}$ ($x_i \notin P_{\mathrm{lt}(\mathcal{I})}(\mathrm{lt}(h))$) induced by non-multiplicative prolongations*
2. *Syzygies $\mathbf{S}_{h,\mathbf{x}^\mu}$, where $\mathbf{x}^\mu \in (\mathrm{lt}(\mathcal{I}) : \mathrm{lm}(h)) \cap \mathbb{K}[P_{\mathrm{lt}(\mathcal{I})}(h)]$ induced by multiplicatively annihilating $\mathrm{lt}(h)$ mod $\mathrm{lt}(\mathcal{I})$*

Choose suitable module term ordering on $(\mathcal{R}/\mathcal{I})^{|H|}$. Then
$H_1 = \{\mathbf{S}_{h,x_i} \mid h \in H\} \cup \{\mathbf{S}_{h,\mathbf{x}^\mu} \mid h \in H\}$ minimal PB of $\mathrm{Syz}^1_{\mathcal{P}/\mathcal{I}}(H)$.

Iteration $\rightsquigarrow$ $\mathcal{R}/\mathcal{I}$-free resolution (not minimal in general)

$$\cdots \to (\mathcal{R}/\mathcal{I})^{|H_1|} \to (\mathcal{R}/\mathcal{I})^{|H|} \to \mathcal{J}/\mathcal{I} \to 0.$$

An Example

Example

$\mathcal{R} = \mathbb{K}[x, y, z]$, $\mathcal{I} = \langle z^3 \rangle$, $\mathcal{J} = \langle xyz, y^2z, yz^2, \mathcal{I} \rangle$.

$$\cdots \to (\mathcal{R}/\mathcal{I})^4 \overset{\varphi_4}{\to} (\mathcal{R}/\mathcal{I})^4 \overset{\varphi_3}{\to} (\mathcal{R}/\mathcal{I})^4 \overset{\varphi_2}{\to} (\mathcal{R}/\mathcal{I})^4 \overset{\varphi_1}{\to} (\mathcal{R}/\mathcal{I})^3 \overset{\varphi_0}{\to} \mathcal{J}/\mathcal{I} \to 0$$

Mappings φ_k given by matrices D_k, with:

$$D_0 = \begin{pmatrix} xyz & y^2z & yz^2 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},$$

$$D_2 = \begin{pmatrix} z & 0 & 0 & 0 \\ -y & z^2 & 0 & 0 \\ x & 0 & z^2 & 0 \\ 0 & xz & yz & z^2 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} z^2 & 0 & 0 & 0 \\ y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},$$

and $D_4 = D_2$.

Minimal Resolutions from Relative Pommaret-like Bases

- $P_{\mathcal{I}}$: *Pommaret-like division relative to a monomial ideal* $\mathcal{I}$; relative Pommaret-like basis exists for ideals in relative quasi-stable position
- Monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is *Clements-Lindström (C-L)*, if it is generated by $\{x_i^{a_i}, x_{i+1}^{a_{i+1}}, \dots, x_n^{a_n}\}$ with $2 \leq a_n \leq a_{n-1} \leq \cdots \leq a_{i+1} \leq a_i$.

Theorem

$\mathcal{I} \trianglelefteq \mathcal{R}$ C-L ideal, $\mathcal{J} \supset \mathcal{I}$ monomial ideal generated by minimal relative Pommaret-like basis $H \subset (\mathcal{J} \setminus \mathcal{I}) \cap \mathcal{T}$. If H is the minimal generating set of $\mathcal{J}$ relative to $\mathcal{I}$ and for each $t \in H$ and $x_a^{p_a} \in \mathrm{NMP}_{P_{\mathcal{I}}}(t, H)$, the unique $P_{\mathcal{I}}$-divisor $s \in H$ of $t \cdot x_a^{p_a}$ fulfils $\mathrm{cls}(s) > \mathrm{cls}(t)$, then the free resolution of $\mathcal{J}$ over $\mathcal{R}/\mathcal{I}$ induced by H is minimal.

Table of Contents

Marked Sets

$\tilde{\mathcal{I}}$ quasi-stable $\rightsquigarrow \mathrm{PB}(\tilde{\mathcal{I}})$ minimal Pommaret basis, $\mathcal{N}(\tilde{\mathcal{I}}) := \mathcal{T} \setminus \tilde{\mathcal{I}}$.
Work with homogeneous polynomials only.

Definition

- *Marked polynomial*: $f \in \mathcal{R}$ having head term $\mathrm{Ht}(f)$ with coefficient 1.
- $\mathrm{PB}(\tilde{\mathcal{I}})$*-marked set*: $F = \{f_\alpha \mid \mathbf{x}^\alpha \in \mathrm{PB}(\tilde{\mathcal{I}})\} \subset \mathcal{R}$ with $\mathrm{Ht}(f_\alpha) = \mathbf{x}^\alpha$ and $\mathrm{supp}(f_\alpha - \mathbf{x}^\alpha) \subset \langle \mathcal{N}(\tilde{\mathcal{I}}) \rangle_{\mathbb{K}}$.

Marked set $F \rightsquigarrow$ noetherian confluent reduction relation $\longrightarrow_{F^*}$ with

$$\mathbf{x}^\rho \longrightarrow_{F^*} \mathbf{x}^\rho - \mathbf{x}^\mu f_\alpha$$

if $\mathbf{x}^\rho$ in Pommaret cone of $\mathbf{x}^\alpha$ and $\mathbf{x}^\rho / \mathbf{x}^\alpha = \mathbf{x}^\mu$.

Marked Bases

Definition

A $\mathrm{PB}(\tilde{\mathcal{I}})$-marked set F is a $\mathrm{PB}(\tilde{\mathcal{I}})$-*marked basis* of the ideal $\langle F \rangle$ if the graded decomposition $\mathcal{R} = \langle F \rangle \oplus \langle \mathcal{N}(\tilde{\mathcal{I}}) \rangle_{\mathbb{K}}$ holds.

Theorem (Ceria et al. $\leq$ 2019)

Let $\tilde{\mathcal{I}} \subseteq \mathcal{R}$ be a quasi-stable monomial ideal, and F a $\mathrm{PB}(\tilde{\mathcal{I}})$-marked set. The following statements are equivalent:

1. *F is a $\mathrm{PB}(\tilde{\mathcal{I}})$-marked basis;*
2. *$f \longrightarrow^{+}_{F^*} 0$, for every $f \in \langle F \rangle$;*
3. *$x_i f_\alpha \longrightarrow^{+}_{F^*} 0$ for every $f_\alpha \in F$ and for every $x_i \in \mathrm{NM}_P(f_\alpha)$.*

Relative Marked Bases

$\tilde{\mathcal{J}} \supseteq \tilde{\mathcal{I}}$ quasi-stable ideals, $\mathcal{I} = \langle F \rangle$ where F is a $\mathrm{PB}(\tilde{\mathcal{I}})$-marked basis,
$H \subset \mathcal{R}$ subset of a $\mathrm{PB}(\tilde{\mathcal{J}})$-marked set, such that
$\mathrm{Ht}(H) = \mathrm{PB}(\tilde{\mathcal{J}}) \setminus \mathrm{PB}(\tilde{\mathcal{I}})$, $\mathcal{J} = \langle F \cup H \rangle \subseteq \mathcal{R}$.

Definition

H $\mathrm{PB}(\tilde{\mathcal{J}})$*-marked basis rel. to* $\mathcal{I}$ if: $\mathcal{R} = \mathcal{I} \oplus (\langle \mathcal{N}(\tilde{\mathcal{I}}) \rangle_{\mathbb{K}} \cap \mathcal{J}) \oplus \langle \mathcal{N}(\tilde{\mathcal{J}}) \rangle_{\mathbb{K}}$.

Theorem

The set H *is a* $\mathrm{PB}(\tilde{\mathcal{J}})$*-marked basis relative to* $\mathcal{I}$ $\Longleftrightarrow$
$\mathcal{J}$ *is generated by a* $\mathrm{PB}(\tilde{\mathcal{J}})$*-marked basis containing* H. $\Longleftrightarrow$
$\mathcal{J}$ *contains a* $\mathrm{PB}(\tilde{\mathcal{J}})$*-marked set* G *containing* H *such that:*

1. $\forall h_\beta \in H, \forall x_i > \mathsf{cls}(\mathrm{Ht}(h_\beta)),\ x_i h_\beta \longrightarrow^{+}_{G^*} 0$
2. $\forall g_\alpha \in G \setminus H, \forall x_i > \mathsf{cls}(\mathrm{Ht}(f_\alpha)),\ x_i g_\alpha \longrightarrow^{+}_{G^*} 0$
3. $\forall f \in F \setminus G,\ f \longrightarrow^{+}_{G^*} 0.$

Open Coverings and Lex Points

$\mathcal{I}$ quasi-stable ideal for which $\mathcal{R}/\mathcal{I}$ is Cohen-Macaulay (e.g., C-L ideal):

Proposition

For a given degree $q \geq 0$, let $F \subset (\mathcal{R}/\mathcal{I})_q$ be a finite set of degree q. Then there exists a transformation $g \in \mathrm{PGL}(k+1)$ such that $\tilde{g}(F)$ is a marked set over a quasi-stable monomial submodule of $\mathcal{R}/\mathcal{I}$.

$\rightsquigarrow$ *open covering* of *Hilbert scheme* of ideals in $\mathcal{R}/\mathcal{I}$ with predefined Hilbert polynomial.

Remark

If $\mathcal{I}$ is Macaulay-Lex (e.g., C-L ideal), can analyse the lex point of such Hilbert schemes using relative marked bases.

Topics for Future Work

- Recursive criteria for relative quasi-stability
- *Resolving decompositions* including non-multiplicative powers
- Explicit differential formulas for quasi-stable ideals in C-L rings
- Investigation of more classes of monomial quotient rings

Thank you for your attention!